Shape Statistics

LETTRES SCIENCES ÉCOLE NORMALE SUPÉRIEURE

ÉCOLE NATIONALE DES PONTS ET CHAUSSÉES

INRIA

Aim : object classification or recognition, by building priors on the shape of the object, considering the information given by a sample set of different examples of this object.

Technical considerations

We use the Level-Set technique, ie we represent a curve by the zero-level of a function defined on the whole plane, and not by a list of points.

Curve Warping

Let A and B be two planar curves. We would like to warp continuously A to B.

Shape space, derivatives, Hausdorff distance

We consider the space $\mathcal{X}$ of all curves A in $\mathcal{C}^2(\mathbb{S}_1, \mathbb{R}^2)$ such that $\|\partial_p A(p)\|$ does not vary with p.

Deformation fields for a given curve A : elementary deformations which one can apply to A.

$\hookrightarrow$ fields δA in $\mathcal{C}^1(\mathbb{S}_1, \mathbb{R}^2)$, with the intuitive warping

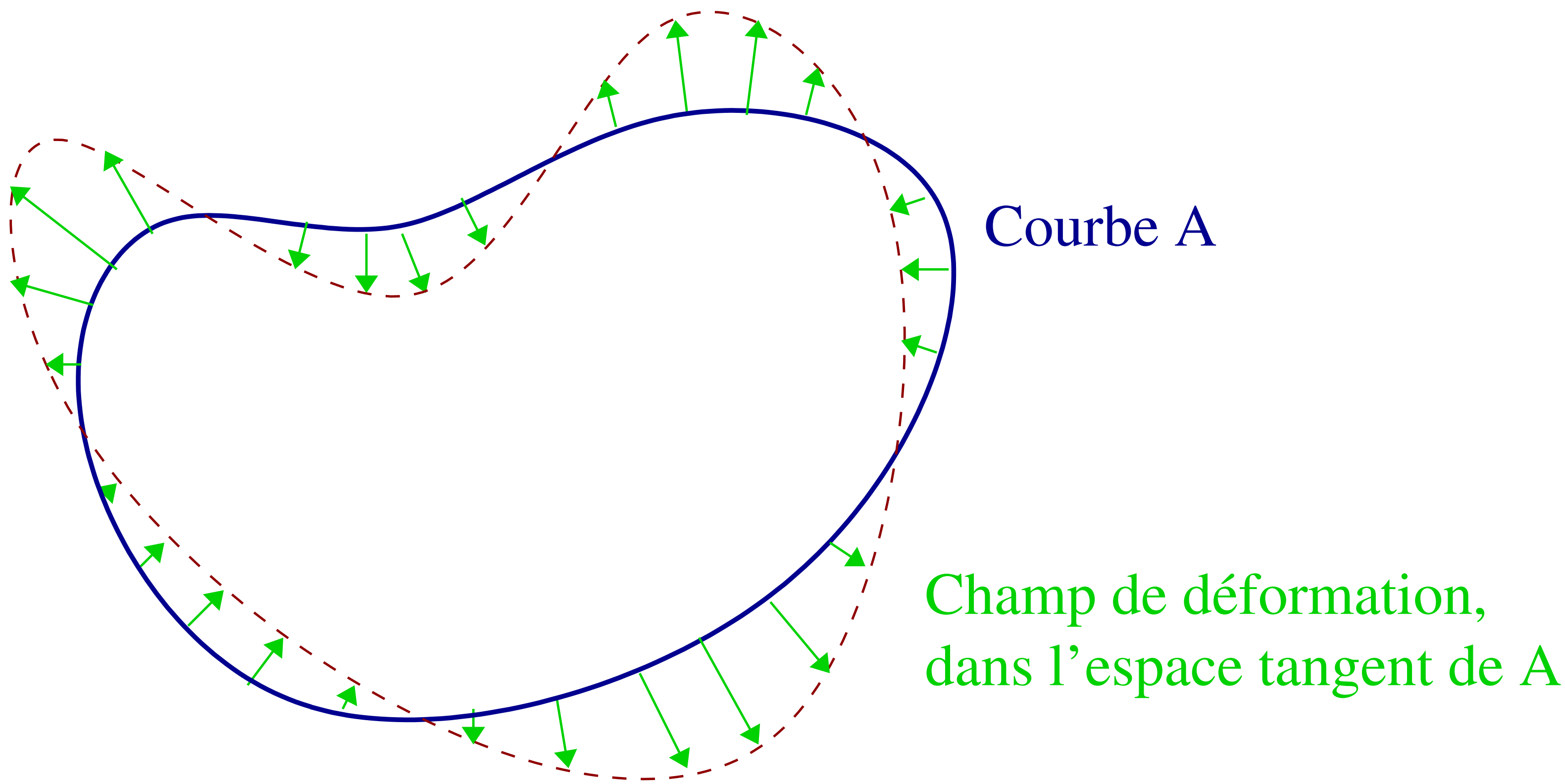
$$A(p) \mapsto A(p) + \delta A(p)$$

$\hookrightarrow$ normal fields β in $\mathcal{C}^1(\mathbb{S}_1, \mathbb{R})$, with the warping

$$A(p) \mapsto A(p) + \beta(p)\, \vec{n}(p)$$

$\hookrightarrow$ scalar product in the tangent space :

$$\langle \beta_1 \,|\beta_2 \rangle = \int_A \beta_1 \beta_2 d\sigma$$


Courbe A
Champ de déformation,
dans l'espace tangent de A

Natural distances

Natural “geodesic distance” :

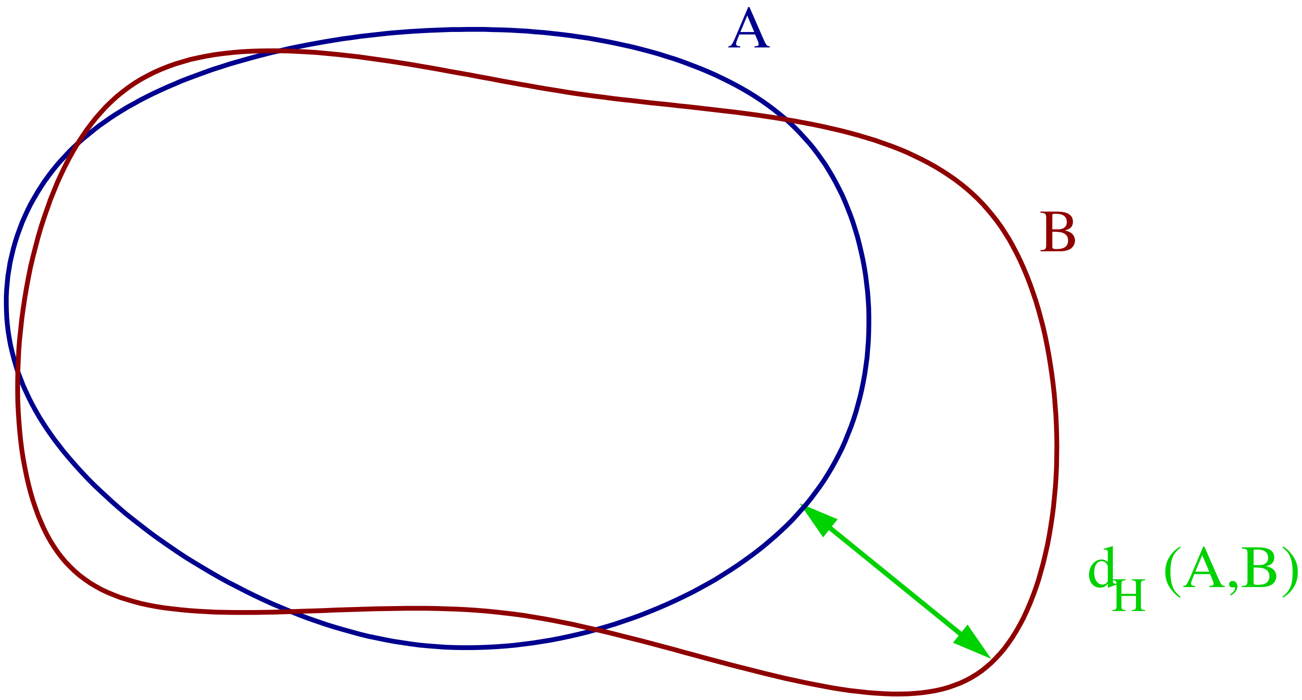
$$d_G(A,B) = \inf_{\Gamma \in \mathcal{F}(A,B)} \int_\Gamma \|\Gamma'(t)\| dt$$

where $\mathcal{F}(A,B)$ is the set of all paths in $\mathcal{C}^1$ going from A to B.

Hausdorff distance :

$$d_H(A,B) = \sup_{x \in A} d(x,B) + \sup_{y \in B} d(y,A)$$

Not derivable...


A
B
dH (A,B)

Derivative of a function E with respect to a curve : linear function L such that, for all deformation field v :

$$Lv = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(E(A + \varepsilon v) - E(A))$$

$\hookrightarrow$ gradient : deformation field $\nabla E(A)$ satisfying, for all v,

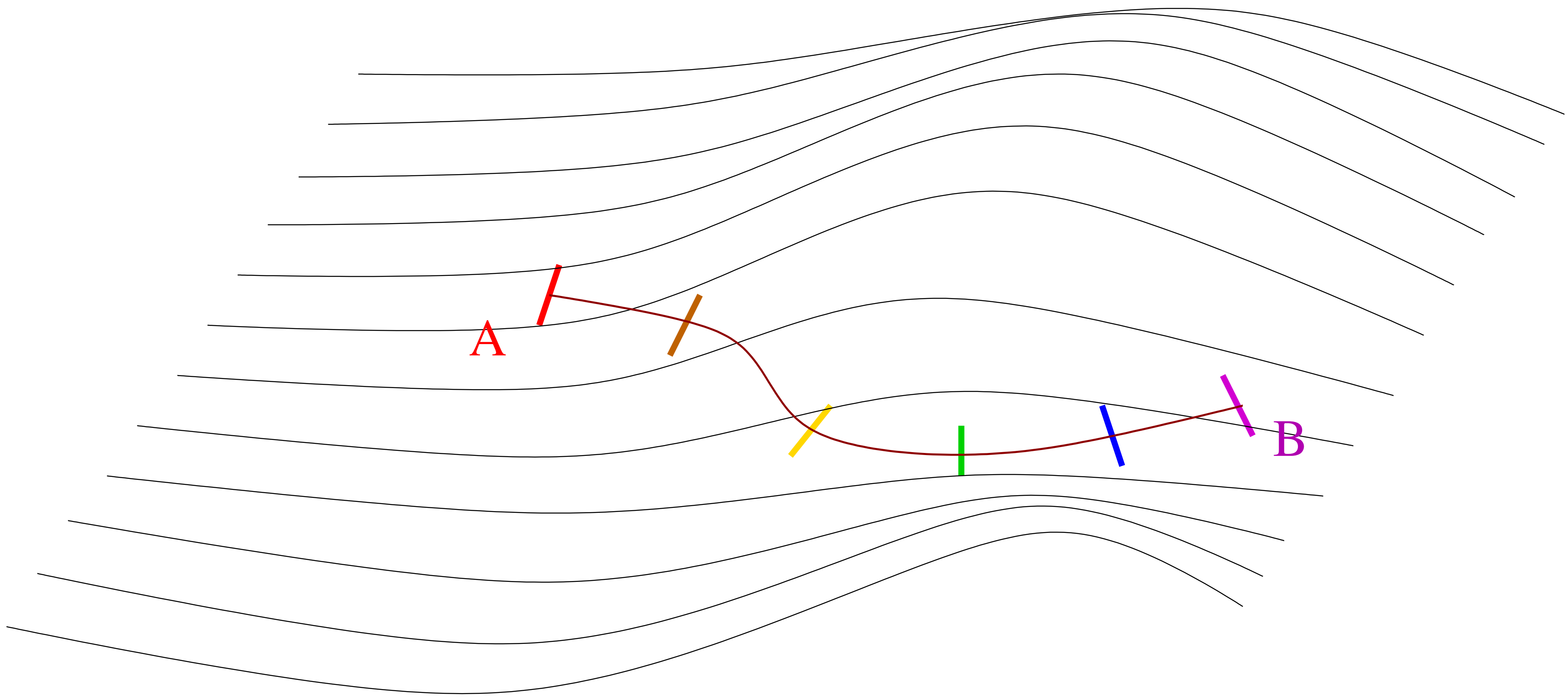
$$Lv = \langle \nabla E(A) \, | v \rangle$$

$\hookrightarrow$ gradient descent on E :

$$\partial_t \mathcal{C} = \nabla E(\mathcal{C}(t))$$

or

$$\partial_t \mathcal{C} = \frac{\nabla E(\mathcal{C}(t))}{\|\nabla E(\mathcal{C}(t))\|}$$


A
B

Smoothing Hausdorff, Notations

We need a smooth function for a gradient descent, near from the Hausdorff distance.

Main idea : if f is a positive continuous function :

$$\lim_{\alpha \to \infty} \left(\int_A (f(y))^\alpha \, dy \right)^{1/\alpha} = \sup_{y \in A} f(y)$$

$\hookrightarrow$ smooth approximation of $\sup_A$:

$$\left(\int_A (f(y))^\alpha \, dy \right)^{1/\alpha}$$

for any real α.

Notation for a mean quantity :

$$\langle f \rangle_A = \frac{1}{|A|} \int_A f$$

$\hookrightarrow$ For a suitable mesurable injective function φ :

$$\langle f \rangle_A^{\varphi} = \varphi^{-1} \left(\frac{1}{|A|} \int_A \varphi \circ f \right)$$

$\hookrightarrow$ if Ψ grows quickly :

$$\langle f \rangle_A^{\Psi} \simeq \sup f$$

$\hookrightarrow$ if φ decreases quickly :

$$\langle f \rangle_A^{\varphi} \simeq \inf f$$

$\hookrightarrow$ approximation of the Hausdorff distance :

$$\max\left(\left\langle \langle d(\cdot,\cdot) \rangle_B^{\varphi} \right\rangle_A^{\Psi},\ \left\langle \langle d(\cdot,\cdot) \rangle_A^{\varphi} \right\rangle_B^{\Psi} \right)$$

Particular case : approximation of the max of two numbers :

$$\langle a, b\rangle^{\Psi} = \Psi^{-1}\left(\frac{1}{2}\Psi(a) + \frac{1}{2}\Psi(b)\right)$$

$\hookrightarrow$

$$E_H(A,B) = \left\langle \left\langle \langle d(\cdot,\cdot)\rangle_B^{\varphi} \right\rangle_A^{\psi}, \left\langle \langle d(\cdot,\cdot)\rangle_A^{\varphi} \right\rangle_B^{\psi} \right\rangle^{\Psi}$$

$\hookrightarrow$ Max of two lengths :

$$E_L(A,B) = \langle |A|, |B| \rangle^{\Psi}$$

$\hookrightarrow$ Hybrid energy :

$$E_M = \langle E_H, \eta E_L \rangle^{\Phi}$$

where η is the ratio of the characteristical lengths in the problem (distance between curves, length difference).

Problems with E_H :

$\hookrightarrow$ not a distance on the space of curves $\mathcal{X}$,

$\hookrightarrow$ $\forall A \in \mathcal{X},\ \ E_H(A, A) \neq 0$,

$\hookrightarrow$ two curves A and B may satisfy $E_H(A, A) > E_H(A, B)$,

$\hookrightarrow$ a gradient descent with respect to C on $E_H(C, B)$ from A does not necessarily end at $C = B$,

$\hookrightarrow$ a gradient descent from B to A does not follow the same path as the one from A to B.

But : E_H is near from d_H : for a suitable choice of functions ψ, Ψ and Φ, we have, for all curves A and B in $\mathcal{X}$:

$$d_H(A,B) - \left(\alpha_\Psi + \alpha_\psi + \Delta_\Phi \frac{|A|+|B|}{2}\right)$$

$$\leqslant$$

$$E_H(A,B)$$

$$\leqslant$$

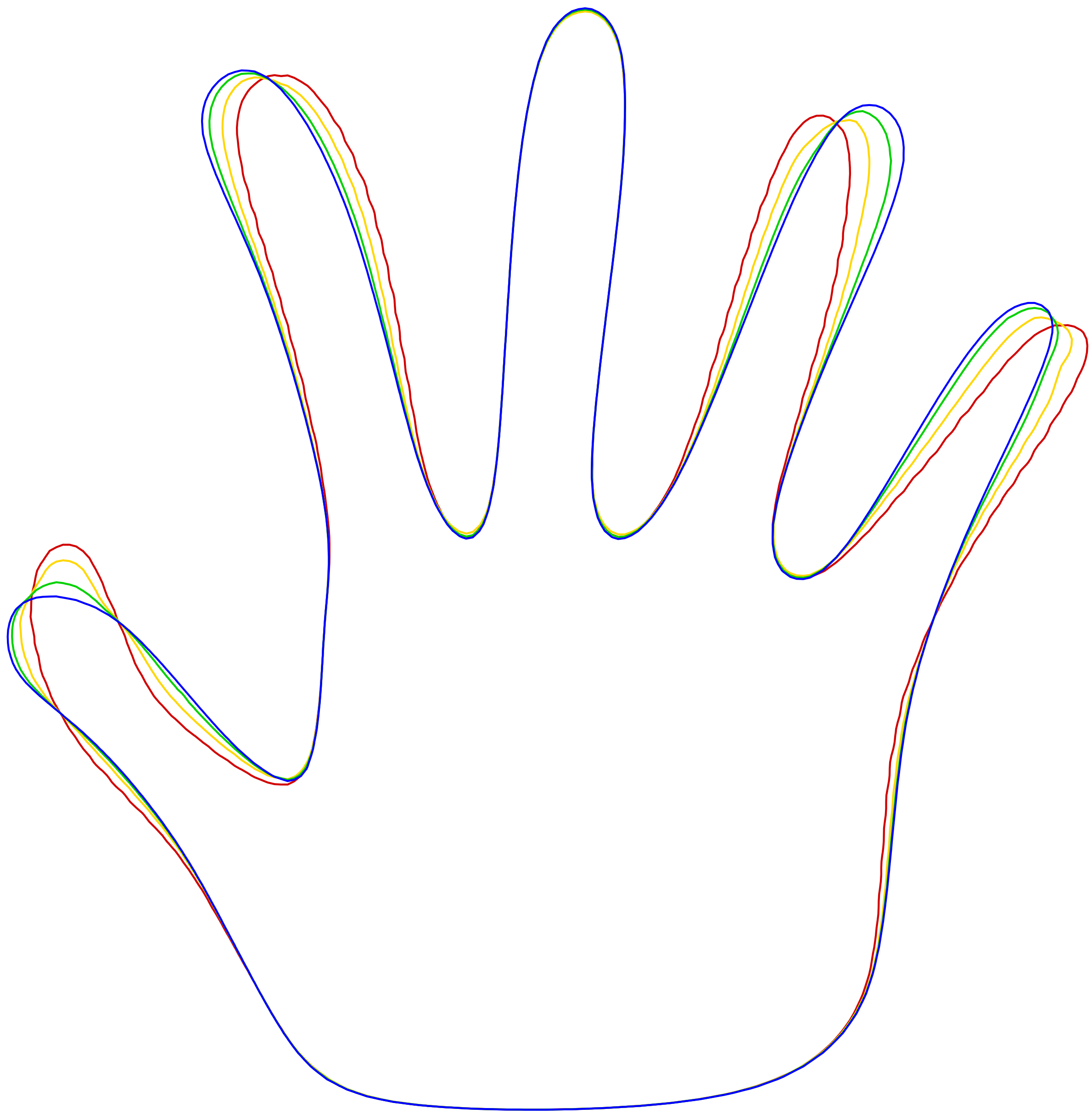
$$d_H(A,B) + \left(\alpha_\Phi + \Delta_\Psi \frac{|A|+|B|}{2}\right)$$

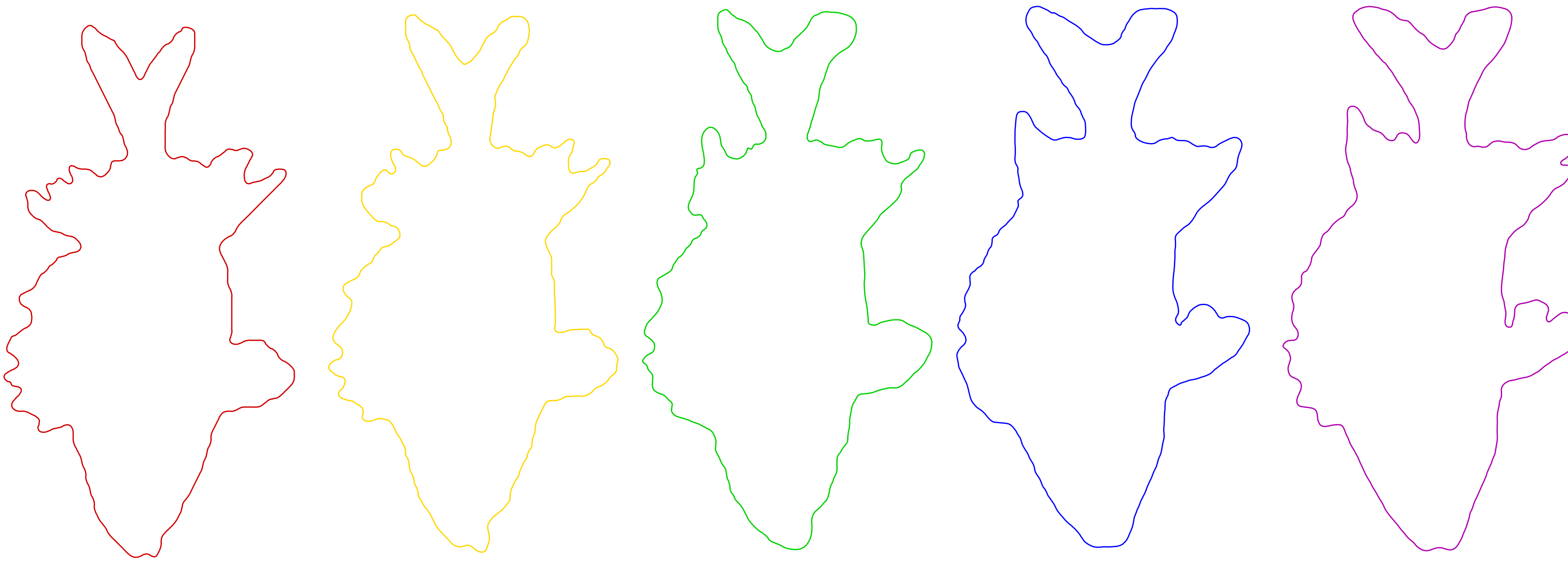
where α_Ψ, α_ψ and Δ_Φ are constants.

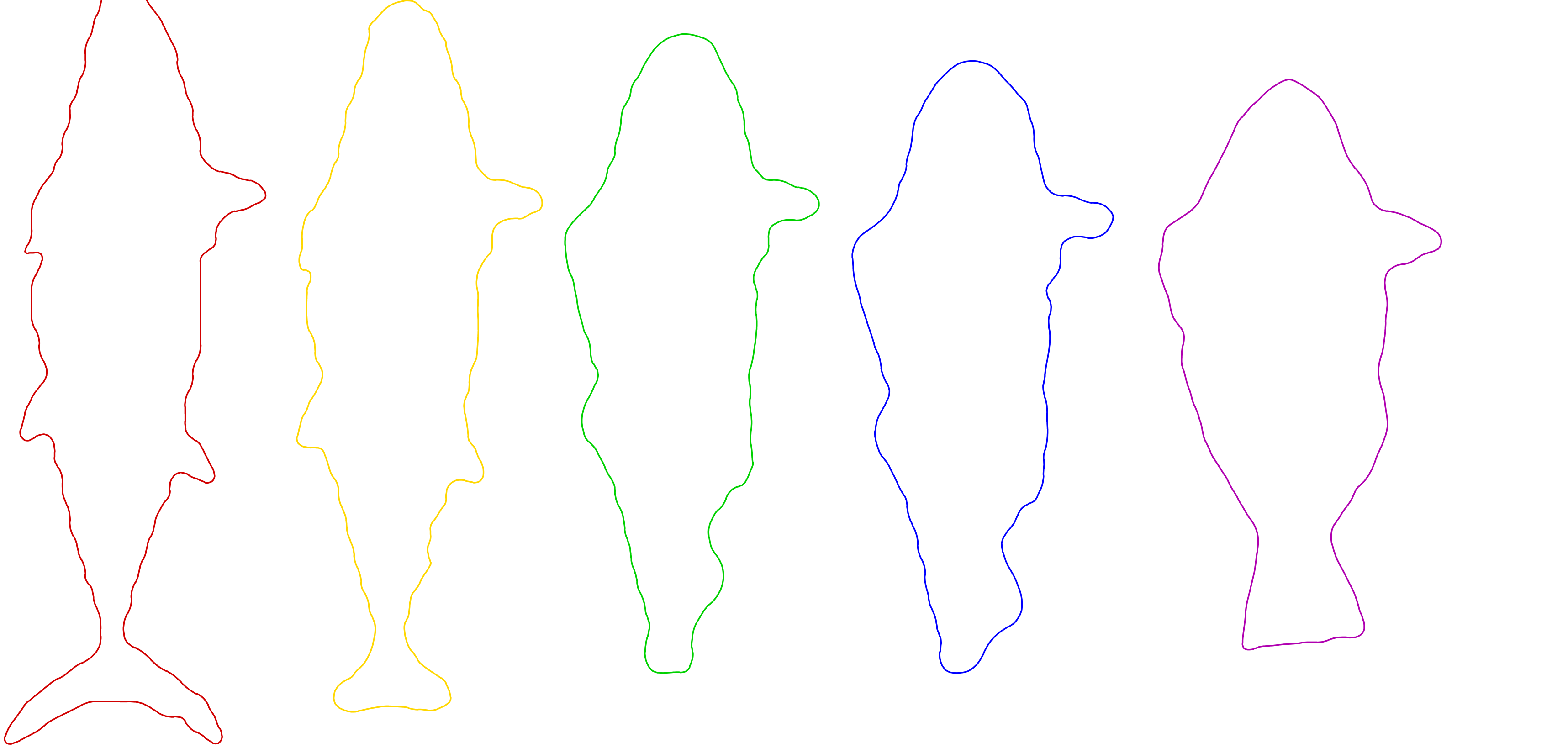
Examples

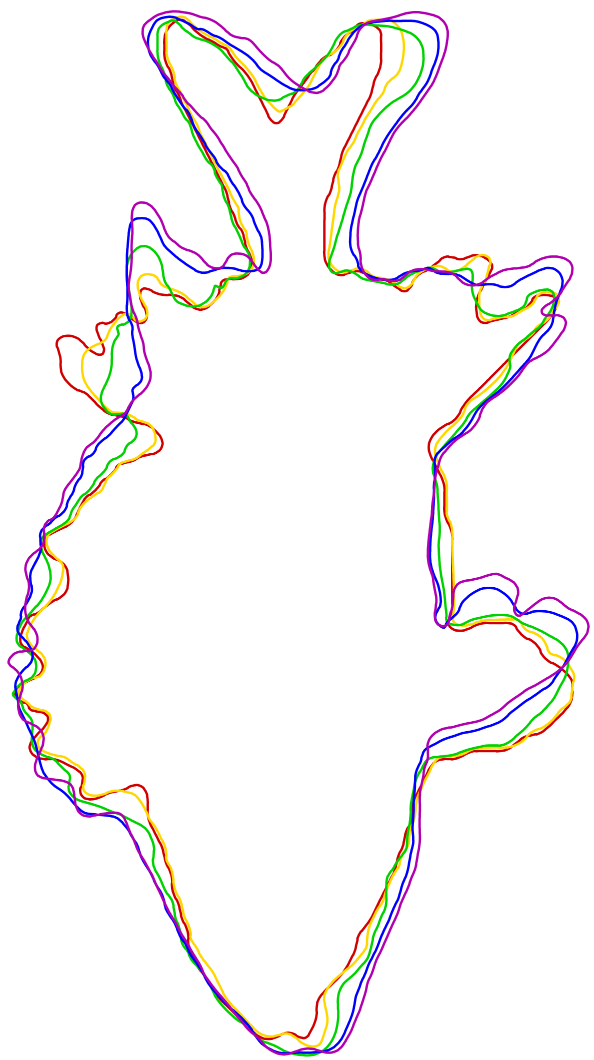
We consider for increasing functions some of the kind $x \mapsto x^\alpha$ and for the decreasing ones $(x + \varepsilon)^{-\alpha}$.

$\hookrightarrow$ we choose α near 4...









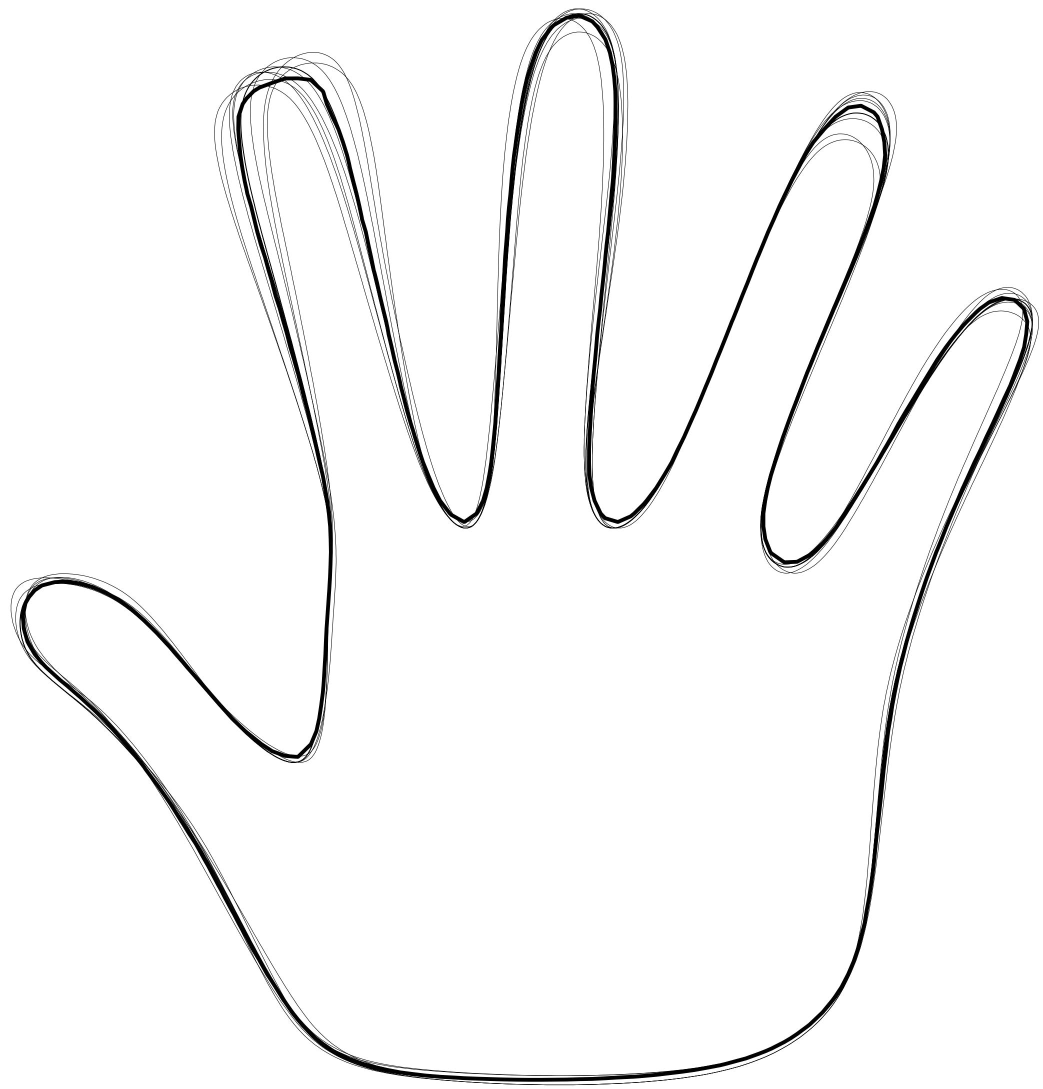
Mean of Curves, Characteristical Deformations

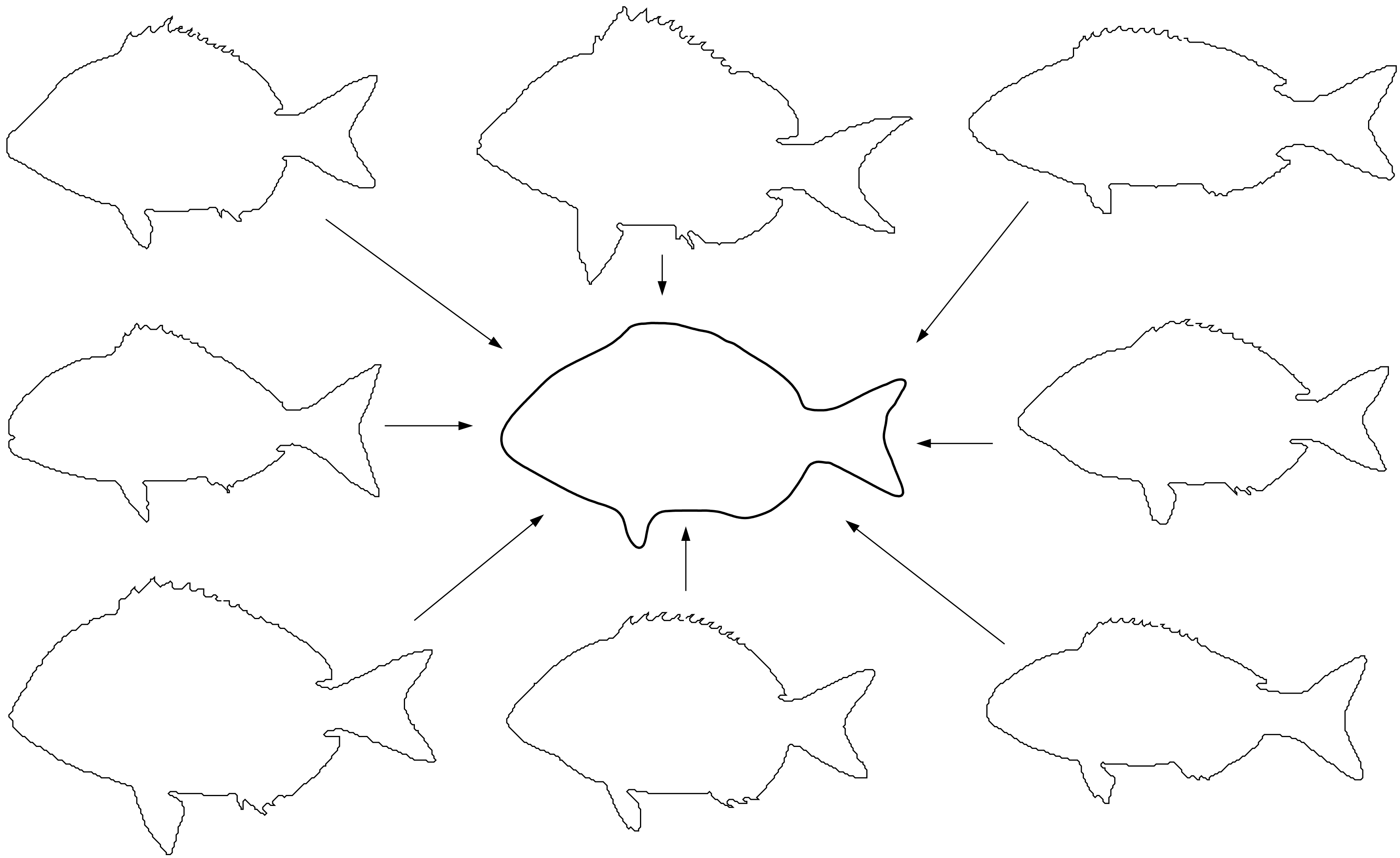
We consider n curves A_i, and search for their mean M :

$\hookrightarrow$ minimize $\sum_i E(M, A_i)^\alpha$, where $\alpha = 1$ or 2,

$\hookrightarrow$ we hope there will not be two many local minima.

$\hookrightarrow$ in practice, it works for reasonable cases.





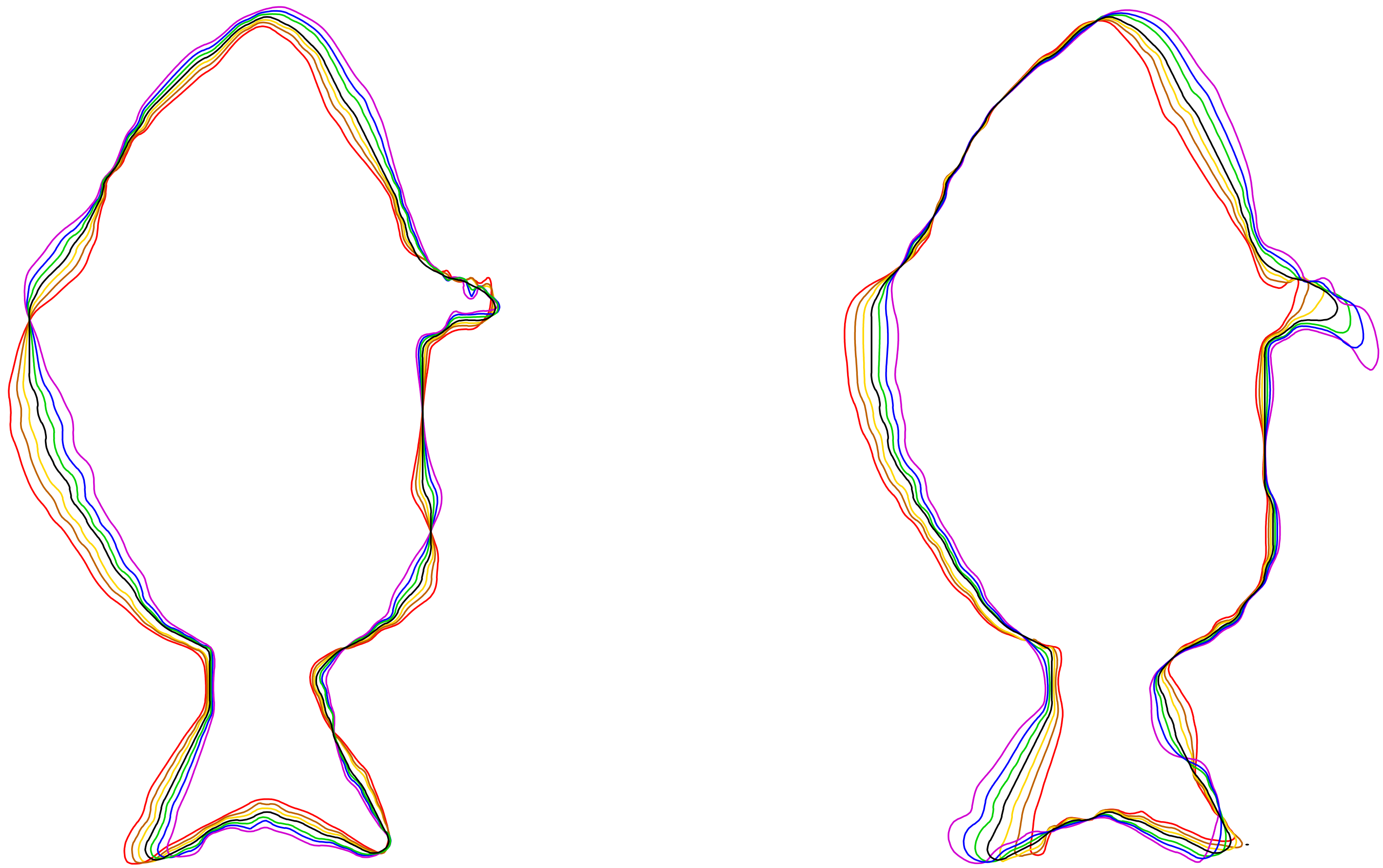
We consider the curves A_i and their mean, M. We would like to define and compute their « characteristical deformations », ie some kind of “standard deviation” but defined for curves.

We note $\delta_i = \nabla_M E(M, A_i)^2$

We build the correlation matrix :

$$\Delta_{i,j} = \left\langle \delta_i \middle| \delta_j \right\rangle$$

and in diagonalizing it, we obtain from the eigenvectors linear combinations of the deformations, hence characteristical deformations β_k.



Standard application : segmentation with priors.

$\hookrightarrow$ knowledge of a mean shape and of the characteristical allowed deformations

Other application : notion of similarity.

We keep the m first modes of deformation. For any curve Y, its distance to the set (curve M + deformations β_k) can be :

$$\sum_{k \leqslant m} \frac{\left\langle \nabla_M E(M,Y)^2 \,\middle|\, \beta_k \right\rangle^2}{\sigma_k^2} + \frac{\left\| R(\nabla_M E(M,Y)^2) \right\|^2}{\sigma_R^2}$$

where

$$R(\beta) = \beta - \sum_{k \leqslant m} \langle \beta | \, \beta_k \rangle \, \beta_k$$

and

$$\sigma_R^2 = \frac{1}{|\mathcal{H}|} \sum_{C \in \mathcal{H}} \| R(C) \|^2$$